Utility maximization for perfect substitutes

Problem Statement

Consider a consumer who derives utility from consuming two goods, x_1 and x_2. The consumer's utility function is given by:

$$U(x_1, x_2) = ax_1 + bx_2, \tag{1}$$

where a and b are positive constants. The consumer has a budget constraint given by:

$$p_1x_1 + p_2x_2 = M, \tag{2}$$

where p_1 and p_2 are the prices of good 1 and good 2, respectively, and M is the consumer's income. Find the optimal consumption bundle and the optimal utility for the consumer.

Solution

Optimal consumption bundle

Since the utility function represents perfect substitutes, the consumer will choose to consume only the good with the highest utility per unit of price, also known as the marginal rate of substitution (MRS). We calculate the MRS for both goods:

$$MRS_1 = \frac{a}{p_1}, \tag{3}$$

$$MRS_2 = \frac{b}{p_2}. \tag{4}$$

Now, we compare the MRS of both goods to determine the optimal consumption bundle:
1. If $MRS_1 > MRS_2$, the consumer will consume only good 1:

$$x_1^* = \frac{M}{p_1}, \quad x_2^* = 0. \tag{5}$$

2. If $MRS_1 < MRS_2$, the consumer will consume only good 2:

$$x_1^* = 0, \quad x_2^* = \frac{M}{p_2}. \tag{6}$$

3. If $MRS_1 = MRS_2$, the consumer is indifferent between the two goods, and any combination of x_1 and x_2 that satisfies the budget constraint will be optimal.

Optimal utility

After finding the optimal consumption bundle, we can calculate the optimal utility by substituting the optimal consumption levels into the utility function:
1. If $MRS_1 > MRS_2$:

$$U^* = a\left(\frac{M}{p_1}\right) + b(0) = \frac{aM}{p_1}. \tag{7}$$

2. If $MRS_1 < MRS_2$:

$$U^* = a(0) + b\left(\frac{M}{p_2}\right) = \frac{bM}{p_2}. \tag{8}$$

3. If $MRS_1 = MRS_2$, the optimal utility will depend on the specific combination of x_1 and x_2 that the consumer chooses, but the total utility will be the same for any combination that exhausts the budget constraint.